UNRAMIFIED EXTENSIONS IN LOCAL FIELDS

CHANGYOON SEUK

ABSTRACT. The study of local fields, especially in finite extensions, exhibits the interplay between residue fields and ramification theory. This paper delves into the exploration of unramified extensions of local fields, focusing on their structure and arithmetic properties.

CONTENTS

1. INTRODUCTION

1.1. **Motivation** In the study of local fields, we may observe the ramified and unramified cases. Briefly, in a field extension $\mathcal{L}$ over $\mathcal{K}$ where A is a Noetherian integrally closed, having $\mathcal{K}$ as a fractional field; $\mathcal{B}$ is a integral closure of $\mathcal{A}$ in $\mathcal{L}$, we see that this $\mathcal{L}/\mathcal{K}$ is unramified when the *ramification index* e=1. In other words, the residue field extension is separable, and that the *valuation* of $\mathcal{K}$ extends to $\mathcal{L}$ without a change in the ramification index. This is perhaps the most general and simple case. For instance, let $\mathcal{K} = \mathbb{Q}_p$ and $\mathcal{L} = (\mathbb{Q}_p(\zeta_p))$ where ζ_p is the p-th root of unity, $\mathcal{K}/\mathcal{L}$ is unramified. However, when the field extension is ramified, we must have the *ramification index* $e > 1$. For example, take $\mathcal{K} = \mathbb{Q}_p$ and $\mathcal{L} = \mathbb{Q}_p(p\sqrt{p})$, then the degree $[\mathcal{L} : \mathcal{K}] = p$ and $e = p > 1$; hence, it is ramified. As this is not only complicated but also requires a sophiscated view, we shall discuss the simpler case, which is unramifed extension at this time.

1.2. **Construction** The first part of this paper is discussing the basic result of the local fields. Mainly definition and properties of *discrete valuation ring* and *Dedekind domains*. Next, we discuss the extension and completion in order to give a general perspectives. Then we delve deep in the section of *discriminant and different* in ramification and prove the main theorem:

Theorem 1.1. *(Main) The extension $\mathcal{L}/\mathcal{K}$ be unramified at the prime ideal $\mathfrak{q} \subset \mathcal{B}$, then it is necessary and sufficient that the prime ideal $\mathfrak{q}$ does not divide the different $\mathfrak{D}_{\mathcal{B}/\mathcal{A}}$.*

Prior to beginning, it is fruitful to review the preliminaries(see [CL21]) from (Mostly) Commutative Algebra by Antoine Chambert-Loir. The rest of sections are from Local Fields by Jean-Pierre Serre(see [Ser80]).

1.3. **Acknowledgement** I would like to give a special appreciation to my advior Professor Amadou Bah, who spent decent amount of time with me for the discussion ovet the summer 2024. This paper, perhaps the whole contruction, would be impossible to craft without Professor Bah's instruction.

2. Preliminaries

Definition 2.1. Ring: $\mathcal{R} = (\mathcal{R}, +, \cdot)$ is a ring satisfying properties: $(\mathcal{R}, +)$ is an abelian group, $(\mathcal{R}, \cdot)$is a monoid, and left and right distributive law holds for any elements in $\mathcal{R}$.

Definition 2.2. Integral Domain: A commutative ring $\mathcal{R}$, that has a unit $(1 \neq 0)$ with no zero divisors. Equivalentely, a nonzero ring $\mathcal{R}$ is an integral domain $\Leftrightarrow \forall x, y \in \mathcal{R}$, with $x \neq 0, y \neq 0$, then $xy \neq 0$ $\Leftrightarrow \forall x, y \in \mathcal{R}$, if $xy = 0$, then either $x = 0$ of $y = 0 \Leftrightarrow$ the cancellation law holds for $\mathcal{R}$.

Definition 2.3. Ideals(First founded by Richard Dedekind, 1871) A subset $\mathcal{I}$ of $\mathcal{R}$ is an ideal if $\mathcal{I}$ has (1):*absorbing property* for all $r \in \mathcal{R},\ s \in \mathcal{I}$ we have $rs \in \mathcal{I}$(left ideal), $sr \in \mathcal{I}$(right ideal). We then write $\mathcal{R}\mathcal{I} \subseteq \mathcal{I}$ and (2):$\mathcal{I}$ is an additive subgroup of $(\mathcal{R}, +)$. In fact, it is a normal subgroup, $\mathcal{I} \lhd \mathcal{R}$ since the coset multiplication is well-defined in the quotient $\mathcal{R}/\mathcal{I}$.

(1) Principal ideal: In a ring $\mathcal{R}$, we have the principal ideal generated by π, denoted $(\pi) = \{s\pi : s \in \mathcal{R}\}$.
(2) Principal ideal domain(PID): Integral domain in which every ideal is principal.
(3) Prime ideal: $\mathcal{I} \subset \mathcal{R}$ is a prime ideal if $\mathcal{I} \neq \mathcal{R}$,and for every $r, s \in \mathcal{R}$, if $r, s \notin \mathcal{I}$ then $rs \notin \mathcal{I}$. Equivalentely, if $rs \in \mathcal{R}$, then either $r \in \mathcal{I}$ or $s \in \mathcal{I}$.
(4) Maximal ideal: $\mathcal{I} \in \mathcal{R}$ be a maximal ideal if $\mathcal{I} \neq \mathcal{R}$ and if $\mathcal{J}$ is an ideal such that $\mathcal{I} \subseteq \mathcal{J}$, then either $\mathcal{I} = \mathcal{J}$ or $\mathcal{J} = \mathcal{R}$.

Remark 2.4. Assume that $\mathcal{R}$ is a commutative ring, then we know that the ideal $\mathfrak{p} \subset \mathcal{R}$ is a prime if and only if the quotient $\mathcal{R}/\mathfrak{p}$ is an integral domain.

Remark 2.5. $\mathcal{R}/\mathcal{I}$ is a field if and only if $\mathcal{I}$ is a maximal ideal. In fact, a maximal ideal is a prime ideal, but the converse is not always true. e.g. Take a look at ring of integers $\mathcal{R} = \mathbb{R}$ with ideal $\mathcal{I} = (0)$. Cleary, we have $\mathbb{R}/(0) \cong \mathbb{Z}$, which is an integral domain, but not a field, hence it is not maximal.

Definition 2.6. Module: Let $\mathcal{R}$ be a commutative ring. A right $\mathcal{R}$-module is a set $\mathcal{M}$ endowed by scaler multiplication and internal addition. That is $\mathcal{M} \times \mathcal{R} \to \mathcal{M}$ defined by $(m, r) \mapsto mr$ for any $m \in \mathcal{M}$ and $r \in \mathcal{R}$. $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ denoted by $(m, n) \mapsto m + n$ for any $m, n \in \mathcal{M}$. Analogously, a letf $\mathcal{R}$-module $\mathcal{M}$ also satisfies internal addition, but the change of order in scalar multiplication, such that $\mathcal{R} \times \mathcal{M} \to \mathcal{M}$ is now $(r, m) \mapsto rm$. In fact, $\mathcal{R}$-module $\mathcal{M}$ is an abelian group satifying the properites of the commutative ring $\mathcal{R}$. Also, in the sense of linear algebra, $\mathcal{R}$-module $\mathcal{M}$ is a $\mathcal{R}$-linear combination of elements in $\mathcal{M}$, such that

$$R\text{-module } \mathcal{M} = r_1 m_1 + \cdots + r_i m_i = \sum_{k=1}^{i} r_k m_k \text{ ,where } r_k \in \mathcal{R},\ m_k \in \mathcal{M}$$

(1) Submodule: From a ring $\mathcal{R}$, we have a $\mathcal{R}$-module $\mathcal{M}$. A $\mathcal{R}$-submodule $\mathcal{N}$ is a subset $\mathcal{N} \subset \mathcal{M}$ satisfying i)$\mathcal{N} \subset$
M is an abelian and ii)for every $r \in \mathcal{R}$ and $m \in \mathcal{M}$, we have $mr \in \mathcal{N}$ and $rm \in \mathcal{N}$.
(2) Morphisms: Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{R}$-modules of a ring $\mathcal{R}$. Then, if we construct a map $f : \mathcal{M} \to \mathcal{N}$, defined by $f(ma + nb) = f(m)a + f(n)b$ for all $a, b, \in \mathcal{R}$ and for all $m, n \in \mathcal{M}$, we call f is a *morphism*. The set of all *morphisms* from $\mathcal{M} \to \mathcal{N}$, we write $Hom_A(\mathcal{M}, \mathcal{N})$. In fact, a *morphism* f of $\mathcal{R}$-modules to be an isomorphism, it is necessary and sufficient that f is a bijection.

3. Discrete Valuation Rings

3.1. **Basic definition and examples.** We first define the key concept of *discrete valuation ring* $\mathcal{A}$, and that is a principal ideal domain(PID) with unique nonzero prime ideal, denoted $\mathfrak{m}(\mathcal{A})$. Notice that $\mathcal{A}/\mathfrak{m}(\mathcal{A})$ is a residue field of $\mathcal{A}$. Since $\mathcal{A}$ is a PID, we have (π) be the nonzero prime ideals, generated by irreducible elements $\pi \in \mathcal{A}$. i.e.$\mathcal{A}$ has exactly one irreducible element up to multiplication; we also call π is a uniformizer. Now, we may see the notion of *valuation*.

Definition 3.1 (Valuation)**.** As above, we find that nonzero ideals are of the form $\mathfrak{m}(\mathcal{A}) = \pi^n \mathcal{A}$. Let $x \in \mathcal{A}\backslash\{0\}$ be arbitray, then we can say $x = \pi^n u$, such that $\pi =$ uniformizer, $u =$ invertible, and $n \in \mathbb{Z}$. Here, the integer n is the *valuation*, denoted $v(x) = n$.

Proposition 3.2 (Explicit)**.** Let $\mathcal{K}$ be the fractional field of $\mathcal{A}$ and $\mathcal{K}^\times$ be the multiplicative group of $\mathcal{K}$. i.e. $\mathcal{K}^\times = \{\frac{a}{b} : a.b \in \mathcal{K}, 0 \neq b, gcd(a, b) = 1\}$. Then we have the following properties.

(1) The valuation map $v : \mathcal{K}^\times \to \mathbb{Z}$ is a surjective group homomorphism.
(2) The weak triangle inequality: $v(x+y) \geq \text{Inf}(v(x), v(y))$ for any $x, y \in \mathcal{K}$

Definition 3.3. Satisfying properties (1) and (2), we get a *discrete valuation ring* $\mathcal{A} = \{v(x) \geq 0 : x \in \mathcal{K}\}$.

Example 3.4. We provide examples for a better understanding of *discrete valuation rings*: (1)Field of formal power series, (2)Noraml algebraic variety, (3)Riemann surface. For a simpler case, let us explain that why a field of *formal power series* is indeed a discrete valuation ring.

Proof. From a field k, let $k((T))$ be the field of *formal power series* in one variable over k. For every non-zero formal series, an element $k((T))$ is of the form:

$$f(T) = \sum_{n \geq n_0} a_n T^n \text{ where } a_{n_0} \neq 0$$

Then, any non-zero element $f(T) \in K((T))$ can be expressed as $f(T) = T^n \cdot g(T)$, where n is a smallest index, such that $a_{n_0} \neq 0$, and $g(T) \in K((T))$ is a unit. This means we can view T^n to π^n with $g(T)$ be a corresponding unit. Hence, we have shown that a *formal power series* is a discrete valuation ring. □

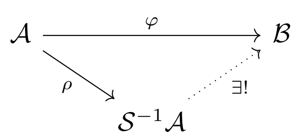
3.2. **Characterizations.** Recall that *Noetherian local ring* is a maximal ideal generated by non-nilpotent element. Then *discrete valuation ring* is necessary and sufficient to be a *Noetherian local ring*. In fact, let $\mathcal{A}$ be a *Noetherian integral domain*, then $\mathcal{A}$ is a *discrete valuation ring* when satisfying the following properties:

(1) $\mathcal{A}$ is integrally closed
(2) $\mathcal{A}$ has a unique non-zero prime ideal

4. Dedekind Domains

The key point here is to remind the localization of prime ideal $\mathfrak{p}$ at $\mathcal{A}$, denoted $\mathcal{A}_\mathfrak{p}$. From a *discrete valuation ring*, we expand toward the larger group called Dedekind domain.

Definition 4.1. Localization: Let $\mathcal{A}$ be a commutative ring(as always it is unital), we take $\mathcal{S}$ be the multiplicative subest of $\mathcal{A}$. i.e. $\mathcal{S}$ is closed under finite producs and $0 \notin \mathcal{S}$. Then, the localization with respect to $\mathcal{S}$ is a ring $\mathcal{S}^{-1}\mathcal{A}$ equipped with a ring homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ in which maps $\mathcal{S}$ into $(\mathcal{S}^{-1}\mathcal{A})^\times$, and satisfying the universal property if $\rho : \mathcal{A} \to \mathcal{B}$ is a ring homomorphism with $\rho(\mathcal{S}) \subseteq \mathcal{B}^\times$. So then, we see the unique ring homomorphism from $\mathcal{S}^{-1}\mathcal{A} \to \mathcal{B}$. In short, we get the commutative map below:



We know that a commutative ring containing 1 is an integral domain, so $\mathcal{A}$ is an integral domain. Let $\mathfrak{p}$ be the prime ideal of A. Then, we have the one-to-one correspondence between the prime ideals of $\mathcal{S}^{-1}\mathcal{A}$ and $\mathcal{A}$ that do not intersect with $\mathcal{S}$ given by the inverse map: $\mathfrak{q} \mapsto \mathfrak{q} \cap \mathcal{A}$ and $\mathfrak{p} \mapsto \mathfrak{p}\mathcal{S}^{-1}\mathcal{A}$. Now, the ring $\mathcal{S}^{-1}\mathcal{A}$ is then denoted $\mathcal{A}_\mathfrak{p}$ with maximal ideal $\mathfrak{p}\mathcal{A}_\mathfrak{p}$; the field of fractions of $A/\mathfrak{p}$ as the residue field. Here, we say $\mathcal{A}_\mathfrak{p}$ is the localization of A at the prime ideal $\mathfrak{p}$.

Definition 4.2. Dedekind domain: *Noetherian integral domain* $\mathcal{A}$ satisfying the two properties below is a Dedekind domain.

(1) For all prime ideals $0 \neq \mathfrak{p} \subset \mathcal{A}$, $\mathcal{A}_\mathfrak{p}$ is a *discrete valuation ring*.
(2) $\mathcal{A}$ is integrally closed, and $\dim(\mathcal{A}) \geq 1$.

Example 4.3. Now, we provide properites and examples of Dedekind domains.

(1) In a *Dedekind domain*, every non-zero fractional ideal is invertible, and this form a group under multiplication.
(2) If $x \in \mathcal{A}$ with x is non-zero, then only finitely many prime ideals contain x.
(3) Every fractional ideal $a \subset \mathcal{A}$ can have a unique factorization of the form: $a = \prod \mathfrak{p}^{v_p(a)}$ where $v_\mathfrak{p}(a)$ are integers almost all zero.
(4) Such examples are the following: every PID is Dedekind; the ring of integers of an algebraic number field is Dedekind; and affine algebraic variety.

5. Extensions and Completion

In this section, we will introduce *separable extensions*; significantly, we may observe the unramified and ramified cases. Then we show the *norm and inclusion* on ideal groups. This will be a foundation toward our main theorem of *unramified extensions* in *discriminant and different*. The key term here is to understand the hypothesis below and expand to the completion of *discrete valuation ring*.

Basic set up: $\mathcal{A}$ is Noetherian integrally closed domain, having $\mathcal{K}$ as field of fractions and $\mathcal{L}$ be the finite extension of $\mathcal{K}$ with the degree $[\mathcal{L}:\mathcal{K}]=n$. Now, $\mathcal{B}$ be the integral closure of $\mathcal{A}$ in $\mathcal{L}$, so that we have $\mathcal{K}\centerdot\mathcal{B}=\mathcal{L}$, and field of fractions of $\mathcal{B}$ is $\mathcal{L}$. From now on, let us denote $\mathcal{AKLB}$ in respect to the basic set up.

Hypothesis 5.1. The ring $\mathcal{B}$ is finitely generated $\mathcal{A}$-module.

Proof. We show this by using the trace map. Let $Tr:\mathcal{L}\to\mathcal{K}$ be the trace map. By linear algebra, we know that such a trace map is non-degenerate bilinear form, so then we see that $Tr(xy)$ is a symmetric non-degenerate $\mathcal{K}$ bilinear form on $\mathcal{L}$. If $x\in\mathcal{B}$ is the conjugated with respect to A, then the conjugates are integral over $\mathcal{A}$. Hence, we find $Tr(x)\in\mathcal{A}$ since $Tr(x)\in\mathcal{K}$. Next, we consider $\{e_i\}$ be a basis in the extenstion $\mathcal{L}/\mathcal{K}$ with $\{e_i\}\in\mathcal{B}$. Let $\mathcal{V}$ be the free $\mathcal{A}-module$ that speend by the basis. Now, for every sub-$\mathcal{A}-module-\mathcal{M}$ of $\mathcal{L}$, let the dual set $\mathcal{M}^*=\{x\in\mathcal{L}|Tr(xy)\in\mathcal{A},\forall y\in\mathcal{M}\}$. Then we observe the inclusion:

$$\mathcal{V}\subset\mathcal{B}\subset\mathcal{B}^*\subset\mathcal{V}^*$$

Notice that the dual $\mathcal{V}^*$ is the free module spanned by the basis dual set $\{e_i\}$. The implies from the Noetherin hypothesis on $\mathcal{A}$; hence, we have shown that $\mathcal{B}$ is finitely generated $\mathcal{A}-module$. □

Having this hypothesis, we find $\mathcal{B}$ is a Noetherian integrally closed domain. In fact, since we already know the definition of *Dedekind domain*, if $\mathcal{A}$ is *Dedekind*, then we also have $\mathcal{B}$ is *Dedekind domain*.

5.1. **Separable extension(basic)** We begin this from the fact that the hypothesis 5.1 is satisfied, so that the finite field extension $\mathcal{L}$ over $\mathcal{K}$ is seperable. Clearly, we have the $\mathcal{AKLB}$ be true. Let $\mathfrak{q}\subset\mathcal{B}$ be the non-zero prime ideal and $\mathfrak{p}=\mathfrak{q}\cap\mathcal{A}$. This implies that the ideal $(\mathfrak{p}\mathcal{B})\subset\mathfrak{q}$ is generated by $\mathfrak{p}$, and this is equivalent to say $\mathfrak{q}$ divides $\mathfrak{p}$, denoted $\mathfrak{q}|\mathfrak{p}$. Next, let us observe $e_{\mathfrak{q}}$ as the exponent of $\mathfrak{q}$ in $\mathfrak{p}\mathcal{B}$; in fact, this $e_{\mathfrak{q}}$ is the *ramification index* of $\mathfrak{q}$ in the extension $\mathcal{L}$ over $\mathcal{K}$. Then we see the prime decomposition:

$$e_{\mathfrak{q}}=v_q(\mathfrak{p}\mathcal{B})\text{ where } v \text{ is the valuation at } \mathfrak{q} \text{ and } \mathfrak{p}\mathcal{B}=\mathfrak{q}_1^{e_1}\times\cdots\times\mathfrak{q}_r^{e_r}=\prod_{\mathfrak{q}|\mathfrak{p}}q^{e_{\mathfrak{q}}} \tag{1}$$

We also have the residue field extension: $\mathcal{B}/\mathfrak{q}$ over $\mathcal{A}/\mathfrak{p}$, if $\mathfrak{q}|\mathfrak{p}$. Here, we denote $f_q=[\mathcal{B}/\mathfrak{q}:\mathcal{A}/\mathfrak{p}]$ be the degree of the residue field extension.

Definition 5.2. Totally ramified: If there is only one prime ideal $\mathfrak{q}$ that divides $\mathfrak{p}$, and the residue degree $f_{\mathfrak{q}}=1$, we say the extenstion $\mathcal{L}/\mathcal{K}$ is *totally ramified* at $\mathfrak{p}$.

Definition 5.3. Unramified: When $e_{\mathfrak{q}}=1$ and the residue field extenstion $\mathcal{B}/\mathfrak{q}$ over $\mathcal{A}/\mathfrak{p}$ is separable, we say $\mathcal{L}/\mathcal{K}$ is *unramified* at $\mathfrak{q}$.

5.2. **The Norm** Let us denote $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$ be the ideal groups of A and $\mathcal{B}$, respectively. We define two homomorphisms in between ideal groups.

$$i:I_{\mathcal{A}}\to I_{\mathcal{B}}\text{ defined by }\mathfrak{p}\mapsto\mathfrak{p}\mathcal{B}=\prod_{\mathfrak{q}|\mathfrak{p}}\mathfrak{q}^{e_{\mathfrak{q}}}$$

and

$$N:I_{\mathcal{B}}\to I_{\mathcal{A}}\text{ defined by }\mathfrak{q}\mapsto\mathfrak{p}^{f_{\mathfrak{q}}}\text{ if }\mathfrak{q}|\mathfrak{p}$$

The two homomorphisms are also known as *Grothendieck groups*, denoted $\mathcal{G}_{\mathcal{A}}$. Suppose now $\mathcal{G}_{\mathcal{A}}$ be the category of $\mathcal{A}$-modules of finite length. Then, if $\mathcal{M}\in\mathcal{G}_{\mathcal{A}}$ and $\mathcal{M}$ has a finite length m, we see the composition series:

$$0=\mathcal{M}_0\subset\mathcal{M}_1\subset\cdots\subset\mathcal{M}_m=\mathcal{M}$$

Observe that each quotient $\mathcal{M}_i/\mathcal{M}_{i-q}\cong\mathcal{A}/\mathfrak{p}_i$, i.e. isomorphic to simple $\mathcal{A}$-module where $\mathfrak{p}_i$ is a non-zero prime ideal of $\mathcal{A}$. So, by the *Jordan-Hölder theorem*, the quotient sequence $\mathcal{A}/\mathfrak{p}_i$ depends only on $\mathcal{M}$; hence, we may put $\chi_{\mathcal{A}}(\mathcal{M})=\prod\mathfrak{p}_i$.
Now, we obtain the following propositions by assuming the lineaerity,

(1) Suppose $\mathcal{M}$ is a $\mathcal{B}$-module of finite length, then $\chi_{\mathcal{A}}(\mathcal{M}) = N(\chi_{\mathcal{B}}(\mathcal{M}))$.
(2) Equivalentely, suppose $\mathcal{M}$ is a $\mathcal{A}$-module of finite length, then $\chi_{\mathcal{B}}(\mathcal{M}_{\mathcal{B}}) = i(\chi_{A}(\mathcal{M}))$.
(3) Suppose now that we restrict our case of N to principal ideals coincides with the usual *norm* map. If $x \in \mathcal{L}$, then $N(x\mathcal{B}) = N_{\mathcal{L}/\mathcal{K}}(x)\mathcal{A}$.

5.3. **Completion** We introduce the definition of *completion* in *discrete valuation rings*. We will apply this notion in the further case of the *discriminant and different*. We first keep in mind that $\mathcal{K}$ is a field on which a discrete valuation v is defined, having a valuation ring $\mathcal{A}$. i.e. the given condition is $v : \mathcal{K}^{\times} \to \mathbb{Z}$ with $\mathcal{A} = \{v(x) \geq 0\}$.

Definition 5.4. Abolute value: Let a be any real number between 0 and 1, then we put

$$\|x\| = a^{v(x)} \text{ for } x \neq 0, \text{ and } \|0\| = 0$$

Then we get the formulas:

$$(1)\|x \cdot y\| = \|x\| \cdot \|y\|;\ (2)\|x+y\| \leq sup(\|x|, \|y\|);\ (3)\|x\| = 0 \text{ if and only if } x = 0$$

Definition 5.5. Completion: Let $\hat{\mathcal{K}}$ be the *completion* if $\mathcal{K}$ for the topology defined by *absolute value* above. Note that the topology does not depend on the choice of a. Since $\hat{\mathcal{K}}$ is also a valued field that extends *absolute value* of $\mathcal{K}$, we have the form:

$$\|x\| = a^{\hat{v}(x)} \text{ for any } x \in \hat{\mathcal{K}}$$

Obeserve that the function $\hat{v}(x)$ is integer-valued; consequently, it is a *discrete valuation* on $\hat{\mathcal{K}}$, whose valuation ring is the closure $\hat{\mathcal{A}}$ of $\mathcal{A}$ in $\hat{\mathcal{K}}$.

6. Ramification

In this section, we give the background of *lattices* prior to *discriminant and different*. The key here is to recall the result of hypothesis 5.1, which is the bilinear form of the trace map. Next, we will use the properties of *discriminant and different* in order to prove our main theorem. Also, we set $\mathcal{A}$ be a Dedekind domain; $\mathcal{K}$ as its field of fractions.

Definition 6.1. Lattices: Let $\mathcal{V}$ be a finite dimensional vector space over $\mathcal{K}$. Then the *lattice* of $\mathcal{V}$ with respect to $\mathcal{A}$ is a sub-$\mathcal{A}$-module of $\mathcal{V}$, which is finitely generated and spans $\mathcal{V}$. Now, X is a free $\mathcal{A}$-module of rank $[\mathcal{V} : \mathcal{K}]$ if $\mathcal{A}$ is principal. Then we can reduce the case by localization; $\mathcal{A}$ with $\mathcal{A}_{\mathfrak{p}}$ and X with $\mathcal{A}_{\mathfrak{p}}X = X_{\mathfrak{p}}$. For example, consider X_1 and X_2 be two *lattices* of $\mathcal{V}$, such that $X_2 \subset X_1$. As we see from the previous section on the homomorphisms of the *norm* map, the invariant $\chi(X_1/X_2)$ is non-zero ideal of $\mathcal{A}$.

Definition 6.2. Discriminant: Let $\mathcal{W} = \Lambda^n\mathcal{V}$ be one-dimensional vector space over $\mathcal{K}$. Then the each *lattice* X of $\mathcal{V}$ associates with a *lattice* of $\mathcal{W}$, this is denoted $X_{\mathcal{W}} = \Lambda^n X$. Since $\mathcal{W}$ is an exterior algebra of $\mathcal{V}$, we easily see that the rank $[\mathcal{W} : \mathcal{K}] = 1$. Suppose now that our $\mathcal{V}$ is provided with a non-degenerate bilinear form $T(x, y)$. In particular, as T extends to $\mathcal{W} = \Lambda^n\mathcal{V}$, we have induced isomorphism: $T : \mathcal{W} \otimes_K \mathcal{W} \to \mathcal{K}$ Here, the $Image(X_{\mathcal{W}} \otimes_{\mathcal{A}} X_{\mathcal{W}})$ under T is a non-zero fractional ideal of K, and this is called the *discriminant* of X with resect to T, denoted $\mathfrak{d}_{X,T}$. If we retrict the case where X is a free $\mathcal{A}$-module with a basis set $\mathcal{S} = \{e_1, \ldots, e_n\}$, then the *discriminant* $\mathfrak{d}_{X,T}$ is the principal ideal generated by the $det(T(e_i, e_j))$.

Remark 6.3. The formula $\mathfrak{d}_{X,T} = det(T(e_i, e_j))$ is the definition of the *discriminant* in such local case.

We define the *different* by assuming that $\mathcal{AKLB}$ be true. Again, recall from the result of the hypothesis 5.1, the trace map $Tr : \mathcal{L} \to \mathcal{K}$ is a surjective homomorphism; the bilinear form $Tr(xy)$ is non-degenerate on the a finite field extension $\mathcal{L}$. Here the *discriminant* of a *lattice* $\mathcal{L}$ is already defined, if this *lattice* is a free $\mathcal{A}$-module with basis set $\{e_i\}$. Explicitly, the *discriminant* is the ideal generated by the determinant of the trace map in basis, such that

$$det(Tr(e_ie_j)) = (det(\sigma(e_i)))^2, \text{ where } \sigma : \{\mathcal{K}\text{-monomorphism of } \mathcal{L}\} \to \{\text{algebraic closure of } \mathcal{K}\}$$

Clearly, we see that $\mathcal{B}$ is a *lattice* of $\mathcal{L}$, and the corresponding *discriminant* is denoted $\mathfrak{d}_{\mathcal{B}/\mathcal{A}}$.

Definition 6.4. Different: Now, we see that $\mathcal{B}$ is a *lattice* of $\mathcal{L}$, and the corresponding *discriminant* is denoted $\mathfrak{d}_{\mathcal{B}/\mathcal{A}}$. Next, let $\mathcal{B}^* = \{y \in \mathcal{L} | Tr(xy) \in \mathcal{A} \text{ and for all } x \in \mathcal{B}\}$ i.e. This is the dual set of $\mathcal{B}$, so the dual *lattice* is denoted $\mathcal{B}_T^*$, and it is called the *codifferent* of $\mathcal{B}$ over $\mathcal{A}$. By taking the inverse of *codifferent*: $(\mathcal{B}_T^*)^{-1}$, we obtatin the *different* $\mathfrak{D}(\mathcal{B}/\mathcal{A})$. As the *different* $\mathfrak{D}(\mathcal{B}/\mathcal{A})$ is a non-zero fractional ideal of $\mathcal{B}$, we take the *norm* to see the following relation:

$$\mathfrak{d}_{X,T} = \chi(\mathcal{B}^*/\mathcal{B}) = N_{\mathcal{L}/\mathcal{K}}(\mathfrak{D}_{\mathcal{B}/\mathcal{A}})$$

6.1. Properties of *discriminant and different*

Proposition 6.5. (Transitivity) Suppose that $\mathcal{M}/\mathcal{L}$ be a separable extension with a finite degree n, and let $\mathcal{C}$ be the integral closure of $\mathcal{A}$ in $\mathcal{M}$. Then, we have the following relations:

$$\mathfrak{D}_{\mathcal{C}/\mathcal{A}} = \mathfrak{D}_{\mathcal{C}/\mathcal{B}} \cdot \mathfrak{D}_{\mathcal{B}/\mathcal{A}} \text{ and } \mathfrak{d}_{\mathcal{C}/\mathcal{A}} = (\mathfrak{d}_{\mathcal{B}/\mathcal{A}})^n \cdot N_{\mathcal{L}/K}(\mathfrak{d}_{\mathcal{C}/\mathcal{B}})$$

Proposition 6.6. (Localization) In respect to the localization of Dedekind domain, suppose that $\mathcal{S}$ be the multiplicative subset of $\mathcal{A}$, then we get the equalities:

$$\mathcal{S}^{-1}\mathfrak{D}_{\mathcal{B}/\mathcal{A}} = \mathfrak{D}_{\mathcal{S}^{-1}\mathcal{B}/\mathcal{S}^{-1}\mathcal{A}} \text{ and } \mathcal{S}^{-1}\mathfrak{d}_{\mathcal{B}/\mathcal{A}} = \mathfrak{d}_{\mathcal{S}^{-1}\mathcal{B}/\mathcal{S}^{-1}\mathcal{A}}$$

Proof. In localization, we know the formula $(\mathcal{S}^{-1}b)^{-1} = \mathcal{S}^{-1}b^{-1}$; so, we show this formula is also appplicalbe in the localization of the dual: $\mathcal{S}^{-1}\mathcal{B}^* = (\mathcal{S}^{-1}\mathcal{B})^*$. In turn, we first show the inclusion: $\mathcal{S}^{-1}\mathcal{B}^* \subset (\mathcal{S}^{-1}\mathcal{B})^*$. Now, let $x = s^{-1}y$ for $s \in \mathcal{S}$ and $y \in \mathcal{B}^*$, then we get the following:

$$Tr(x) = s^{-1}Tr(y) \in \mathcal{S}^{-1}\mathcal{A}$$

Notice that $\mathcal{S}^{-1}\mathcal{B} \subset \mathcal{S}^{-1}\mathcal{B}^*$ and since this dual localization is also a fractional ideal, we have shown initial inclusion. Conversely, suppose $\{b_i\}$ be basis set of $\mathcal{B}$ as an $\mathcal{A}$-module, and let $x \in (\mathcal{S}^{-1}\mathcal{B})^*$. Then, by the trace map, such that $Tr(xb_i) = s^{-1}a_i$ where $a_i \in \mathcal{A}$ and $sx \in \mathcal{B}^*$, we find $Tr(xb_i) \in \mathcal{S}^{-1}\mathcal{A}$. Hence, we have shown that $(\mathcal{S}^{-1}\mathcal{B})^* \subset \mathcal{S}-1\mathcal{B}^*$; thus, we obtain the desired equality. ☐

Proposition 6.7. (Completion) Let $\mathfrak{q} \subset \mathcal{B}$ be a prime ideal and let $\mathfrak{p} = \mathfrak{q} \cap \mathcal{A}$. Suppose that in the completion $\hat{\mathcal{B}}_\mathfrak{q}$, we have the ideal $\hat{\mathfrak{D}}_q$ generated by the *different* $\mathfrak{D}_{\mathcal{B}/\mathcal{A}}$. Then, $\hat{\mathcal{B}}_q$ is the *different* of the ring $\hat{\mathcal{B}}_\mathfrak{q}$ with respect to the ring $\hat{\mathcal{A}}_\mathfrak{p}$.

Proof. Followed by the result of localization in *discriminant and different*, we consider the case when the given Dedekind domain $\mathcal{A}$ is a *discrete valuation ring*. Let $\hat{\mathcal{A}}$ and $\hat{\mathcal{K}}$ be the completion of $\mathcal{A}$ and the fractional field $\mathcal{K}$, respectively. Similarly, let $\{\mathfrak{q}_i\}_{i\in I}$ be the set of prime ideals of $\mathcal{B}$ over $\mathfrak{p}$, we denote $\hat{\mathcal{B}}_i$ be the completion of $\mathcal{B}$ for the valuation defined by $\mathfrak{q}_i$. By the same analogy, we also have $\hat{\mathcal{L}}_i$ for $\mathcal{L}$. Then, we first obeserve the $\hat{\mathcal{K}}$-algebra, such that $\mathcal{L} \otimes_\mathcal{K} \hat{\mathcal{K}} = \hat{\mathcal{L}}$; equivalently, $\hat{\mathcal{A}}$-*lattice* of $\hat{\mathcal{L}}$ is $\hat{\mathcal{B}} = \mathcal{B} \otimes_\mathcal{A} \hat{\mathcal{A}}$. Next, as we have the non-degenerate bilinear form $Tr(xy)$ on $\mathcal{L}$, we take the inverse of the *different*: $(\mathfrak{D}_{\mathcal{B}/\mathcal{A}})^{-1} = \mathcal{B}^*$, and obtain the basis of the dual *lattice* $(\hat{\mathcal{B}})^*$. This means, we have $(\hat{\mathcal{B}})^* = \mathcal{B}^* \otimes_\mathcal{A} \hat{\mathcal{A}}$. On the other hand, we already know by the extension and completion from the previous section that $\mathcal{L} \otimes_\mathcal{K} \hat{\mathcal{K}} = \prod_{i\in I} \hat{\mathcal{L}}_i$ and $\mathcal{B} \otimes_\mathcal{A} = \prod_{i\in I} \hat{\mathcal{B}}_i$. Let Tr_i be the trace of each extension $\hat{\mathcal{L}}_i/\hat{\mathcal{K}}$, then we see that the bilinear form

$$Tr(xy) = \bigoplus_{i\in I} Tr_i(xy) \text{ on } \hat{\mathcal{L}}_i$$

This implies to

$$\left(\prod_{i\in I} B_i\right)^* = \prod_{i\in I} (B_i)^*$$

In which shows that $(\hat{\mathcal{B}})^* = \prod_{i\in I}(\hat{B}_i^*)$. Clearly, since the *codifferent* of $\mathcal{B}$ with respect to $\mathcal{A}$ generates each of the completion $\hat{\mathcal{L}}_i$, we take the inverses to get the *different*, as desired. ☐

Theorem 6.8. *(Unramified Extensions) The given condition is same as the properties of discriminant and different, and that is let $\mathfrak{q}$ be a prime ideal of $\mathcal{B}$ with $\mathfrak{p} = \mathfrak{q} \cap \mathcal{A}$. In order to have the extension $\mathcal{L}/K$ is unramified at $\mathfrak{q}$, it is necessary and sufficient that $\mathfrak{q}$ does not divide the different $\mathfrak{D}_{\mathcal{B}/\mathcal{A}}$.*

Proof. The results from the properties of *discriminant and different*, especially on the *localization and completion*, allow us to reduce the case when $\mathcal{A}$ is a complete discrete valuation ring. Now, we denote $\mathfrak{K}$ be the residue field of $\mathcal{A}$. As we already know the if $\mathcal{A}$ is Dedekind domain then $\mathcal{B}$ is also Dedekind, we have $\mathcal{B}$ be a *discrete valuation ring* too. Again, by the previous section, where we introduce the general *unramified/totally ramified* cases, we must have the *ramification index* $e_{\mathfrak{q}} = 1$, i.e., $\mathcal{B}/\mathfrak{p}\mathcal{B}$ is a finite separable extension of $\mathfrak{K}$. Recall from the remark 6.3 of the definition of the *discriminant*, we view our case in the local case. So if we have $\{x_i\}$ as a basis of $\mathcal{B}$ over $\mathcal{A}$, then its *discriminant* $\mathfrak{d}_{\mathcal{B}/\mathcal{A}}$ is the principal ideal generated by $d = det(Tr(x_i, x_j))$. Then, we must have $\mathfrak{p}$ does not divide d that the $Image(\bar{d}) \in \mathfrak{k}$ is non-zero. Finally, let $\bar{\mathcal{B}} = \mathcal{B}/\mathfrak{p}\mathcal{B}$, then we get $\bar{d}$ be the *discriminant* of the basis, and the set of images: $\{Imagex_i\}$ form a basis of $\bar{\mathcal{B}}$ over $\mathfrak{K}$. Notice that having $\bar{d}$ be non-zero means $\mathcal{B}/\mathfrak{p}\mathcal{B} = \bar{\mathcal{B}}$ is a seperable $\mathfrak{K}$-algebra, and since we are discusing in the local case, $\bar{\mathcal{B}}$ is obviously a local ring; hence, it is a field. Thus, we have shown that $\mathcal{B}/\mathfrak{p}\mathcal{B}$ is a field with a separable extension of $\mathfrak{K}$. □

References

[CL21] Antoine Chambert-Loir. *(Mostly) Commutative Algebra*, volume 1 of *CBMS Regional Conference Series in Mathematics*. Springer Cham, 2021.

[Ser80] Jean-Pierre Serre. *Local Fields*, volume 67 of *Graduate Texts in Mathematics*. Springer New York, NY, 1980.

Department of mathematics, Columbia univesity
Email address: cs4056@columbia.edu